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ABSTRACT

We derive a set of ordinary differential equations relating the rms emittance of a beam with its nonlinear field energy. The equations are valid in constant or periodic focusing and 1-D, 2-D, or 3-D and thus provide a general framework to study emittance growth. They allow us to estimate the expected total emittance growth, if the change of nonlinear field energy can be predicted from general principles, like the homogenization of charge density in beams that are space-charge dominated. Structure resonances in periodic focusing can be identified as a resonance between the envelope modulation and the oscillating nonlinear field-energy term. The equations lead to estimates for the maximum longitudinal-transverse emittance transfer in situations where equipartitioning occurs. A comparison with results from computer simulation will be the subject of another paper in preparation.

I. INTRODUCTION

Recently, an equation has been presented by Wangler et al.[1] relating the rms emittance growth of a beam to its electric field energy in the case of an azimuthally symmetric charge distribution and a continuous focusing force. (A similar equation has been derived previously by Lapostolle.[2]) The equation by Wangler has been found useful for interpreting the change of rms emittance in numerical simulation under extreme space-charge conditions. In this limiting case, it has been possible to predict final emittance growth from the field energy stored in an initially nonuniform charge distribution.[1] Wangler et al. also confirm the approximate formula for emittance growth, which was previously derived heuristically by Struckmeier, Klabunde, and Reiser[3] assuming a balance between field energy and (transverse) kinetic energy.

Real beams are considered in periodic quadrupole channels and thus deviate from azimuthal symmetry. Moreover, they can be bunched longitudinally and allow for transfer of emittance between the transverse and longitudinal directions, as was shown previously by Jameson[4] and by Hofmann and Bozsik[5] using computer simulation. The question has been raised whether it is possible to generalize Wangler's equation to a realistic multidimensional case with periodic focusing and thus obtain relatively simple equations for emittance growth that can be useful for practical design and to enhance our general understanding of emittance growth. Here we show that it is indeed possible to derive such equations by forming moments of Vlasov's equation (Sec. II). In Sec. III, we specify the general equation to beams in one, two, or three dimensions; in Sec. IV, we present a minimum energy principle; in Sec. V, we derive Debye shielding; and in Sec. VI, we discuss some practical applications.

II. DERIVATION OF BASIC EQUATION

We assume equations of motion in x, y, z with linear, time-dependent external focusing forces and arbitrary space-charge forces, where time is replaced by the distance s (= v • t) and q is the charge of the particles:

$$
\begin{aligned}
x'' + k_x(s)x - \frac{q}{m\gamma^3 v^2} E_x(x,y,z,s) &= 0 \\
y'' + k_y(s)y - \frac{q}{m\gamma^3 v^2} E_y(x,y,z,s) &= 0 \qquad (1) \\
z'' + k_z(s)z - \frac{q}{m\gamma^3 v^2} E_z(x,y,z,s) &= 0 \quad .
\end{aligned}
$$

$\underline{E}$ follows from Poisson's equation

$$\underline{\nabla} \bullet \underline{E} = \frac{q}{\varepsilon_0} \, n(x,y,z,s) \quad , \qquad (2)$$

where n is the density, which is determined by projecting a distribution in 6-D phase space into real space:

$$n = \iiint f(x,y,z,x',y',z',s) \, dx'dy'dz' \quad ,$$

with f satisfying Vlasov's equation ($\underline{v} \equiv d\underline{x}/ds$)

$$\frac{\partial f}{\partial s} + (\underline{x}' \cdot \underline{\nabla})\, f - \left(\underline{k} - \frac{q}{m\gamma^3 v^2}\, \underline{E}\right) \cdot \underline{\nabla_{x'}} f = 0 \quad . \tag{3}$$

We define second-order moments (with N the total number of particles)

$$\overline{x^2} = N^{-1} \int \ldots \int x^2 f \; dx \ldots dz' \quad ,$$

$$\overline{xx'} = N^{-1} \int \ldots \int xx' f \; dx \ldots dz' \quad , \quad \text{etc.,}$$

and derive from Eq. (3) by computing the respective moments:

$$\frac{d}{ds}\, \overline{x^2} - 2\overline{xx'} = 0 \quad ,$$

$$\frac{d}{ds}\, \overline{xx'} - \overline{x'^2} + k_x\, \overline{x^2} - \frac{q}{m\gamma^3 v^2}\, \overline{xE_x} = 0 \quad , \tag{4}$$

$$\frac{d}{ds}\, \overline{x'^2} + 2k_x \overline{xx'} - \frac{2q}{m\gamma^3 v^2}\, \overline{x'E_x} = 0 \quad ,$$

and analogous in y, z.

We define the rms envelope

$$\tilde{x} \equiv \left(\overline{x^2}\right)^{1/2}$$

and rms emittance

$$\varepsilon_x \equiv 4\left(\overline{x^2}\; \overline{x'^2} - \overline{xx'}^2\right)^{1/2} \quad . \tag{5}$$

We follow the procedure by Sacherer[6] and obtain the rms envelope equations

$$\frac{d^2}{ds^2}\, \tilde{x} + k_x(s)\, \tilde{x} - \frac{\varepsilon_x^2(s)}{16\tilde{x}^3} - \frac{q}{m\gamma^3 v^2}\, \frac{\overline{xE_x}}{\tilde{x}} = 0 \qquad \text{(similar in } y,z) \tag{6}$$

by eliminating $\overline{x'^2}$ and $\overline{xx'}$ and ignoring the third part of Eq. (4) in favor of Eq. (5). We observe that neither Eq. (4) nor Eq. (6) is a closed set of equations because, in general, higher-order moments of Vlasov's equation are contained in $\overline{xE_x}$, $\overline{x'E_x}$, and we obtain an infinite set of coupled moment equations. Hence, Eq. (6) contains $\varepsilon_x^2(s)$ as an unknown function, whose derivative can be readily derived from Eq. (5) (similar in y, z):

$$\frac{d}{ds}\,\varepsilon_x^2 = \frac{32q}{m\gamma^3 v^2}\left(\overline{x^2}\ \overline{x'E_x} - \overline{xx'}\ \overline{xE_x}\right) \quad ,$$

$$\frac{d}{ds}\,\varepsilon_y^2 = \frac{32q}{m\gamma^3 v^2}\left(\overline{y^2}\ \overline{y'E_y} - \overline{yy'}\ \overline{yE_y}\right) \quad , \tag{7}$$

and

$$\frac{d}{ds}\,\varepsilon_z^2 = \frac{32q}{m\gamma^3 v^2}\left(\overline{z^2}\ \overline{z'E_z} - \overline{zz'}\ \overline{zE_z}\right) \quad .$$

The difficulty now lies with the unknown moments involving $\underline{E}$. In the following, we attempt to replace these by the electric field energy as a quantity that promises more physical insight. To this end we require that

$$\overline{x'E_x} = N^{-1} \int\ldots\int x'E_x f\ dx\ldots dz' = N^{-1}\iiint E_x n\overline{v_x}\ dx\ dy\ dz \quad , \tag{8}$$

with $n\overline{v_x} \equiv \iiint x'\ f\ dx'dy'dz'$, the local averaged velocity of beam particles (in the moving frame). With the local current given by

$$\underline{j} = q\ n\ v\ \underline{\overline{v}} \quad , \tag{9}$$

we obtain

$$\overline{x'E_x} = (Nqv)^{-1} \iiint E_x j_x\ dx\ dy\ dz \tag{10}$$

and similar for y, z. By integrating Eq. (3), we derive the continuity equation

$$\frac{\partial n}{\partial s} + (qv)^{-1}\, \underline{\nabla} \bullet \underline{j} = 0 \tag{11}$$

and write ($\underline{E} = -\underline{\nabla}\phi$):

$$\iiint \underline{E} \bullet \underline{j}\, dx\, dy\, dz = \iiint \phi\, \underline{\nabla} \bullet \underline{j}\, dx\, dy\, dz = -\, qv \iiint \phi \frac{\partial n}{\partial s}\, dx\, dy\, dz \quad . \tag{12}$$

The integration is performed over a volume V, which contains the beam in its interior; hence, we may neglect a surface integral. Using Poisson's equation, we obtain

$$\iiint \underline{E} \bullet \underline{j}\, dx\, dy\, dz = -\, v \frac{d}{ds} W - \varepsilon_o v \iint_S \phi \frac{\partial}{\partial s} E_n\, d\sigma \quad , \tag{13}$$

where E_n is the normal component of E on the surface S of our integration volume and

$$W = \frac{\varepsilon_o}{2} \iiint_V \underline{E}^2\, dx\, dy\, dz \tag{14}$$

is the field energy within V.

Using Eqs. (10) and (13), we can add the three equations, Eq. (7), after dividing them by $\overline{x^2}$, $\overline{y^2}$, and $\overline{z^2}$, respectively. We thus obtain, with the first equation of Eq. (4):

$$\frac{1}{\overline{x^2}} \frac{d}{ds} \varepsilon_x^2 + \frac{1}{\overline{y^2}} \frac{d}{ds} \varepsilon_y^2 + \frac{1}{\overline{z^2}} \frac{d}{ds} \varepsilon_z^2 =$$

$$\frac{32q}{m\gamma^3 v^2} \left[-\frac{1}{Nq} \frac{dW}{ds} - \frac{\varepsilon_o}{Nq} \iint_S \phi \frac{\partial E_n}{\partial s}\, d\sigma - \frac{1}{2} \left(\frac{1}{\overline{x^2}} \frac{d\overline{x^2}}{ds}\, \overline{xE_x} + \frac{1}{\overline{y^2}} \frac{d\overline{y^2}}{ds}\, \overline{yE_y} + \frac{1}{\overline{z^2}} \frac{d\overline{z^2}}{ds}\, \overline{zE_z} \right) \right] . \tag{15}$$

We observe that no approximation has been made so far in deriving Eq. (15), which relates the coupled change of the three emittances to the change of field energy. The remaining term on the right-hand side will be shown below to give, with some approximations, the field energy of a uniform density beam.

We remark that Eq. (15) immediately yields the respective 2-D equation for transverse beam dynamics of a long beam (ε_z, E_z equal to zero); it also yields the 1-D case describing purely longitudinal dynamics (ε_x, ε_y, E_x, and E_y equal to zero). The right-hand side of Eq. (15) has to be evaluated separately for each dimension (Sec. III).

It is also useful to derive an energy principle from the third equation, Eq. (4), for x, y, and z. By adding them and using Eqs. (10) and (13), we obtain

$$\frac{d}{ds}\left(\overline{x'^2} + \overline{y'^2} + \overline{z'^2}\right) + \left[k_x(s)\,\frac{d}{ds}\,\overline{x^2} + k_y(s)\,\frac{d}{ds}\,\overline{y^2} + k_z(s)\,\frac{d}{ds}\,\overline{z^2}\right]$$

$$+ \frac{2}{m\gamma^3 v^2 N}\left(\frac{d}{ds}\,W + \varepsilon_o \iint_s \phi\,\frac{\partial}{\partial s}\,E_n\,d\sigma\right) = 0 \quad . \tag{16}$$

For constant focusing, we can write this as total energy conservation law

$$T + V_{ex} + \frac{1}{\gamma^2 N}\,W = \text{const} \tag{17}$$

with

$$T \equiv \frac{m\gamma v^2}{2}\,\left(\overline{x'^2} + \overline{y'^2} + \overline{z'^2}\right) \tag{18}$$

(kinetic energy in beam frame)

$$V_{ex} \equiv \frac{m\gamma v^2}{2}\left(k_x\overline{x^2} + k_y\overline{y^2} + k_z\overline{z^2}\right) \tag{19}$$

(potential energy caused by external focusing forces).

Here we have neglected the boundary integral, which is justified if the boundary is far away and thus $dE_n/ds \rightarrow 0$.

III. GENERALIZED EMITTANCE EQUATIONS IN DIFFERENT DIMENSIONS

A. Three-Dimensional Equation

To relate the third term on the right-hand side of Eq. (15) to the field energy of a uniformly filled ellipsoid, we have to calculate its potential and field energy. In App. A, we derive the potential of a uniform rotationally symmetric ellipsoid with semiaxes a (in x, y) and c (in z) and show that the field energy inside a large sphere of radius R is given by

$$W_u = \frac{N^2 q^2}{40\pi\varepsilon_o}\left[\frac{6}{c}\left(1 - f + \frac{c^2}{a^2} f\right) - \frac{5}{R}\right] , \tag{20}$$

where f(c/a) is a geometry factor (= 1/3 for a spherical bunch). By allowing for a and c to vary with s, we can calculate the time derivative of W_u and find (App. A)

$$\frac{dW_u}{ds} = - \frac{3N^2 q^2}{20\pi\varepsilon_o}\left[\frac{dc}{ds}\frac{f}{a^2} + \frac{da}{ds}\frac{1 - f}{ac}\right] . \tag{21}$$

To show the relation of dW_u/ds with the remaining term in Eq. (15), we define

$$I \equiv \frac{1}{2}\left(\frac{1}{\overline{x^2}}\frac{d\overline{x^2}}{ds}\,\overline{xE_x} + \frac{1}{\overline{y^2}}\frac{d\overline{y^2}}{ds}\,\overline{yE_y} + \frac{1}{\overline{z^2}}\frac{d\overline{z^2}}{ds}\,\overline{zE_z}\right) . \tag{22}$$

For a uniformly charged ellipsoid, we calculate I from the potential and find readily

$$I_u = - \frac{1}{Nq}\frac{dW_u}{ds} . \tag{23}$$

For a more general ellipsoid, it can be shown[6] that the averaged quantities $\overline{xE_x}$, $\overline{yE_y}$, and $\overline{zE_z}$ depend only weakly on the actual charge distribution if bunches with identical rms dimensions (rms equivalent bunches) and ellipsoidal symmetry are compared. The latter requires a particle density of the kind

$$n(x,y,z,s) = n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} , s\right) \tag{24}$$

for which one finds[6]

$$\overline{xE_x} = \overline{yE_y} = \frac{3Nq}{20\sqrt{5}\pi\varepsilon_o \overline{z^2}^{1/2}} \frac{1-f}{2} \lambda_3 \tag{25}$$

and

$$\overline{zE_z} = \frac{3Nq}{20\sqrt{5}\pi\varepsilon_o \overline{z^2}^{1/2}} \frac{\overline{z^2}}{\overline{x^2}} f \lambda_3 \tag{26}$$

with λ_3 depending on the profile in Eq. (24):

1	uniform
$\frac{25\sqrt{5}}{21\sqrt{7}} \approx 1.006$	parabolic
$\frac{5\sqrt{5}}{6\sqrt{\pi}} \approx 1.051$	Gaussian
$\frac{25}{8\sqrt{3\pi}} \approx 1.018$	hollow (r^2 • Gaussian)

(Note that our λ_3 corresponds to $5\sqrt{5}\ \lambda_3$ in Ref. 6).

We thus can generally write

$$I = -\frac{1}{Nq} \lambda_3 \frac{dW_u}{ds} \tag{27}$$

and obtain from Eq. (15) the 3-D generalized emittance equation (here actually proven under the constraint of $\varepsilon_x = \varepsilon_y$ and $\overline{x^2} = \overline{y^2}$):

$$\frac{1}{\overline{x^2}} \frac{d}{ds} \varepsilon_x^2 + \frac{1}{\overline{y^2}} \frac{d}{ds} \varepsilon_y^2 + \frac{1}{\overline{z^2}} \frac{d}{ds} \varepsilon_z^2 = -\frac{32}{m\gamma^3 v^2 N} \left(\frac{dW}{ds} - \lambda_3 \frac{dW_u}{ds} \right) . \tag{28}$$

The integration in W is performed over a large enough volume so that $dE_n/ds \to 0$; hence, we may neglect the surface integrals.

Keeping in mind that the right-hand side vanishes for a uniformly charged bunch, we have thus shown that the change of emittance is directly related to the change of the nonlinear field energy. We observe that we have on the left-hand side the weighted sum of the $d\varepsilon^2/ds$ for x, y, and z, as a result of the coupling introduced by the space-charge force. This equation, therefore, promises an estimate of the change of the weighted sum of emittances if the change in nonlinear field energy can be estimated from general principles. It then allows also an estimate of the maximum emittance transfer if "equipartitioning" should occur as a result of a coherent instability[5] or single-particle resonance. The actual dynamics of coherent instabilities requires solving equations for higher than second-order moments, which is beyond the framework of our derivation. Equation (28) also indicates the possibility of slowly growing emittance caused by "structure resonances" in a periodic focusing system if W oscillates with a period close to the focusing period. This can happen for certain values of the phase advance σ_o and σ (see Ref. 7 for the 2-D analogue).

We thus suggest that Eq. (28) gives a rather general framework to describe emittance growth. Its practical value depends yet on the possibility of estimating changes of the nonlinear field energy, without actually calculating its time dependence, which is in fact possible in many situations.

Equation (28) is supplemented by the rms envelope equations derived by Sacherer.[6] With Eqs. (6), (25), and (26), we obtain for the rotationally symmetric ellipsoid

$$\frac{d^2}{ds^2}\tilde{x} + k_x(s)\,\tilde{x} - \frac{e_x^2(s)}{16\tilde{x}^3} - \frac{Nq^2}{20\sqrt{5}\pi\varepsilon_o m\gamma^3 v^2 \tilde{x}\tilde{z}}\, h_x\!\left(\frac{\tilde{z}}{\tilde{x}}\right) = 0 \tag{29}$$

and

$$\frac{d^2}{ds^2}\tilde{z} + k_z(s)\,\tilde{z} - \frac{\varepsilon_z^2(s)}{16\tilde{z}^3} - \frac{Nq^2}{20\sqrt{5}\pi\varepsilon_o m\gamma^3 v^2 \tilde{z}\tilde{x}}\, h_z\!\left(\frac{\tilde{z}}{\tilde{x}}\right) = 0 \tag{30}$$

with

$$\tilde{x} \equiv \overline{x^2}^{\,1/2} \qquad (= a/\sqrt{5} \text{ for uniform ellipsoid}) \quad ,$$

$$\tilde{z} \equiv \overline{z^2}^{\,1/2} \qquad (= c/\sqrt{5} \text{ for uniform ellipsoid}) \quad ,$$

$$h_x\left(\frac{\tilde{z}}{\tilde{x}}\right) = \frac{3}{2}\left[1 - f\left(\frac{\tilde{z}}{\tilde{x}}\right)\right]\lambda_3 \quad , \tag{31}$$

and

$$h_z\left(\frac{\tilde{z}}{\tilde{x}}\right) = 3\,\frac{\tilde{z}}{\tilde{x}}\,f\left(\frac{\tilde{z}}{\tilde{x}}\right)\lambda_3 \quad . \tag{32}$$

Note that for a spherical bunch, we have $f = 1/3$; hence, $h_x = h_z = \lambda_3 \approx 1$. For a nearly spherical bunch ($0.8 \lesssim \tilde{z}/\tilde{x} \lesssim 5$), we have approximately, with $f \approx 1/(3\tilde{z}/\tilde{x})$(see App. A),

$$h_x \approx 1 + \frac{1}{2}\left(1 - \frac{\tilde{x}}{\tilde{z}}\right) \tag{33}$$

and

$$h_z \approx 1 \quad . \tag{34}$$

B. Two-Dimensional Equation

In App. B, we show that for a continuous beam with uniform elliptic cross section, the field energy calculated within a large circle of radius R is given by

$$W_u = \frac{N^2q^2}{16\pi\varepsilon_o}\left(1 + 4\,\ln\frac{2R}{a + b}\right) \quad , \tag{35}$$

where a and b are the semiaxes in x and y. Thus, we find

$$\frac{dW_u}{ds} = -\,4w_o\,\frac{\frac{d}{ds}(a + b)}{a + b} \quad , \tag{36}$$

where we have introduced

$$w_o \equiv \frac{N^2q^2}{16\pi\varepsilon_o} \tag{37}$$

as field energy within the actual beam volume. The 2-D equivalent of I in Eq. (22) is readily seen to obey again

$$I_u = -\,\frac{1}{Nq}\,\frac{dW_u}{ds} \quad . \tag{38}$$

The main difference with the 3-D case is that $\overline{xE_x}$ and $\overline{yE_y}$ are independent of the density profile, as long as elliptical symmetry is satisfied[6]:

$$n(x,y,s) = n\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}, s\right) \quad . \tag{39}$$

We then have $I = I_u$ and obtain (again neglecting a surface integral, if R is sufficiently large)

$$\frac{1}{\overline{x^2}} \frac{d}{ds} \varepsilon_x^2 + \frac{1}{\overline{y^2}} \frac{d}{ds} \varepsilon_y^2 = - \frac{32}{m\gamma^3 v^2 N} \frac{d}{ds} (W - W_u) \quad . \tag{40}$$

It is appropriate to introduce the in-beam field energy w_o as the normalization constant and rewrite the 2-D generalized emittance equation as

$$\frac{1}{\overline{x^2}} \frac{d}{ds} \varepsilon_x^2 + \frac{1}{\overline{y^2}} \varepsilon_y^2 = - 4\, K \frac{d}{ds} \frac{W - W_u}{w_o} \quad . \tag{41}$$

Note that $\frac{d}{ds} W_u$ has to be replaced by $\lambda_2 \frac{d}{ds} W_u$, with λ_2 a correction factor close to unity if Eq. (39) is not satisfied.* Here we have introduced the generalized perveance given by (following the notation of Ref. 1)

$$K \equiv \frac{Nq^2}{2\pi\varepsilon_o m\gamma^3 v^2} \quad . \tag{42}$$

For a round beam $\left(\overline{x^2} = \overline{y^2} \text{ and } \varepsilon_x = \varepsilon_y\right)$, we readily obtain

$$\frac{d}{ds} \varepsilon_x^2 = - 2\, K\, \overline{x^2} \frac{d}{ds} \frac{W - W_u}{w_o} \quad , \tag{43}$$

which agrees with the equation derived in Ref. 1 for continuous focusing (note that $\overline{x^2} = X^2/4$, with X the radius of an equivalent uniform beam as used in Ref. 1).

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*In this case, numerical calculations (private communication, P. M. Lapostolle, 1985) show that λ_2 differs very little from unity, for instance, a few times 10^{-3} for a rectangular cross-section beam. In long periodic systems, variation of λ_2 might have a bearing on slow emittance growth in the same fashion as the resonant growth from a periodically varying $W - W_u$ as discussed in Sec. VI.3.

Equation (41) is again a promising tool to describe changing emittances if the change of the nonlinear field energy is known or can be estimated from some general principles. Note that $(W - W_u)/w_o$ only depends on the type of non-uniform density, but not the actual size of the beam. For a parabolic density profile, we show in App. D that it has the value 0.0224, whether the cross sections are spherical or elliptic. A Gaussian profile yields 0.154 for a round beam[1] (numerical results indicate the same value for an elliptic cross section).

For completeness, we give the rms envelope equations following from Eq. (6):

$$\frac{d^2}{ds^2}\tilde{x} + k_x(s)\,\tilde{x} - \frac{\varepsilon_x^2}{16\tilde{x}^3} - \frac{K}{2}\frac{1}{\tilde{x}+\tilde{y}} = 0 \tag{44}$$

and

$$\frac{d^2}{ds^2}\tilde{y} + k_y(s)\,\tilde{y} - \frac{\varepsilon_y^2}{16\tilde{y}^3} - \frac{K}{2}\frac{1}{\tilde{x}+\tilde{y}} = 0 \quad . \tag{45}$$

C. One-Dimensional Equation

For a uniformly charged sheet ($|z| \leqq c$ and infinitely extended in x,y), the field energy within $|z| \lesssim L$ is found as in App. C (N particles per unit area in x,y):

$$W_u = \frac{N^2q^2}{\varepsilon_o}\left(\frac{L}{4} - \frac{c}{6}\right) \tag{46}$$

and

$$\frac{dW_u}{ds} = -\frac{N^2q^2}{6\varepsilon_o}\frac{dc}{ds} = -NqI_u \quad . \tag{47}$$

In Ref. 6, $\overline{zE_z}$ has been evaluated to be slightly dependent on the density profile; hence (with $\overline{z^2} = c^2/3$ for a uniform sheet),

$$\overline{zE_z} = \frac{Nq}{2\sqrt{3}\varepsilon_o}\,\overline{z^2}^{\,1/2} \bullet \lambda_1 \quad , \tag{48}$$

where λ_1 is given by (differing by $\sqrt{3}$ from Ref. 6)

1	uniform
0.996	parabolic
0.977	Gaussian
0.987	hollow (z^2 • Gaussian) .

We thus have

$$I = -\frac{1}{Nq}\lambda_1 \frac{dW_u}{ds} \tag{49}$$

and the 1-D generalized emittance equation:

$$\frac{1}{\overline{z^2}}\frac{d}{ds}\varepsilon_z^2 = -\frac{32}{m\gamma^3 v^2 N}\left(\frac{dW}{ds} - \lambda_1 \frac{dW_u}{ds}\right) . \tag{50}$$

The corresponding envelope equation is $\left(\tilde{z} = \overline{z^2}^{\,1/2}\right)$

$$\frac{d^2}{ds^2}\tilde{z} + k_z(s)\,\tilde{z} - \frac{\varepsilon_z^2(s)}{16\tilde{z}^3} - K\frac{\pi}{\sqrt{3}}\lambda_1 = 0 . \tag{51}$$

IV. MINIMUM FIELD ENERGY FOR UNIFORM-DENSITY BEAM

Practical evaluation of the generalized emittance equations requires estimates on the possible change of nonlinear field energy. Numerical calculations for a continuous, round beam have indicated that a uniform-density profile has lower field energy than a variety of peaked or hollow profiles with the same rms radius.[1,3] Here we show that this is generally true in any dimension. Let us thus consider a 3-D bunched beam with the field energy W given by Eq. (14). We require that the variation of

$$S \equiv W + \alpha_1\overline{x^2} + \alpha_2\overline{y^2} + \alpha_3\overline{z^2} \tag{52}$$

be zero, with α_i Lagrange multipliers to keep the rms dimensions constant. Hence,

$$\delta S = \iiint_V \left[\varepsilon_0 \underline{E}\,\delta\underline{E} + N^{-1}\left(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2\right)\delta n\right] dx\,dy\,dz = 0 . \tag{53}$$

By partial integration we can write this as

$$\delta S = \iiint_V \left[\phi + N^{-1}\left(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2\right)\right] \delta n \; dx\, dy\, dz + \varepsilon_0 \iint_S \phi \; \delta E_n d\sigma = 0 \; . \qquad (54)$$

The boundary integral can be neglected for a large enough integration volume because the total charge is kept constant. The variation of density, $\delta n(x,y,z)$, cannot be arbitrary because N must be constant and $n + \delta n \geqq 0$ everywhere. We thus define it by an arbitrary displacement $[\delta x(x,y,z), \delta y(x,y,z), \delta z(x,y,z)]$ of the position vector:

$$\delta n(x,y,z) = n(x + \delta x,\ y + \delta y,\ z + \delta z) - n(x,y,z) = \underline{\nabla} n \bullet \delta \underline{x} \qquad (55)$$

and obtain

$$\delta S = \iiint \left[\phi + N^{-1}\left(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2\right)\right](\underline{\nabla} n \bullet \delta \underline{x}) \; dx\, dy\, dz = 0 \quad , \qquad (56)$$

which is satisfied by either $\phi = - N^{-1}(\alpha_1 x^2 + \alpha_2 y^2 + \alpha_3 z^2)$ (interior of beam) or $n = \text{const} = 0$ (outside) corresponding to a uniformly charged ellipsoid.

The same proof holds for 2-D and 1-D beams in full space, or with a far-away boundary. The important conclusion from this is that relaxation of a distribution with nonuniform density toward one of uniform density is accompanied by a growth of emittance to compensate for the reduced nonlinear field energy.

V. SHIELDING NEAR THE SPACE-CHARGE LIMIT

For a practical evaluation of the generalized emittance equations, we will find it useful to study analytically the nonlinear field energy of stationary distributions in high-current beams. This will enable us to relate the nonlinear field energy of a matched beam to its tune depression ν/ν_0 (or σ/σ_0 in periodic focusing), as a dimensionless parameter describing intensity. For small ν/ν_0, we expect the well-known "plasma-shielding" effect, where the external potential is shielded from the beam interior by the space-charge potential. This collective behavior of an intense beam leads to the development of a practically uniform density for $\nu/\nu_0 \to 0$, regardless of the shape of the distribution function, provided that the external focusing force is linear. In computer simulation, the general observation has been that an intense beam relaxes to a more uniform self-consistent density profile if injected with a nonmatched profile. The relaxation is accompanied by a change of emittance, which we intend to calculate from our theory.

For the sake of simplicity, in the following discussion we assume round (spherical or cylindrical) beams with time-independent (continuous) focusing force.

A. Three-Dimensional Case

We assume, first, a Gaussian distribution function of the single-particle Hamiltonian (thus automatically a stationary distribution):

$$f = \frac{n(0)}{(2\pi\mu)^{3/2}} \exp - \left[\left(\frac{x'^2 + y'^2 + z'^2}{2} + \frac{k}{2} r^2 + \frac{q}{m\gamma^3 v^2} \phi\right)\Big/\mu\right] , \tag{57}$$

where ϕ follows from Poisson's equation obtained by calculating the density

$$\nabla^2\phi = - \frac{q}{\varepsilon_o} n(0) \exp\left[-\left(\frac{k}{2} r^2 + \frac{q}{m\gamma^3 v^2} \phi\right)\Big/\mu\right] \tag{58}$$

with $n(0)$ the density on axis, and $m = \overline{x'^2}$ is the average of x'^2 as a measure for the beam temperature. Equation (58) can be solved explicitly in the low-current limit, where ϕ in the density expression is negligible. Here we are interested in the high-current limit and assume that the nonlinear dependence on ϕ can be expanded as a power series:

$$\nabla^2\phi = - \frac{q}{\varepsilon_o} n(0) \left[1 - \left(\frac{k}{2} r^2 + \frac{q}{m\gamma^3 v^2} \phi\right)\Big/\mu + \ldots\right] , \tag{59}$$

where we retain only the first-order term in the total potential. We make the substitution

$$\tilde{\phi} = \frac{q}{\varepsilon_o} n(0) \left[6 \frac{k}{2\mu} + C\left(\frac{kr^2}{2\mu} - 1 + \frac{q}{m\gamma^3 v^2 \mu} \phi\right)\right] \tag{60}$$

with

$$C \equiv \frac{q^2 n(0)}{\varepsilon_o m\gamma^3 v^2 \mu} \tag{61}$$

and obtain the familiar equation

$$\nabla^2 \tilde{\phi} = C \tilde{\phi} \quad , \tag{62}$$

which is solved by the modified spherical Bessel function $\sqrt{\frac{\pi}{2}/z}\ I_{n+1/2}$ for $n = 0$, and assuming $z \equiv \sqrt{C}\ r$, then

$$\sqrt{\frac{\pi}{2}/z}\ I_{1/2} = z^{-1} \sinh z \quad . \tag{63}$$

Using Eq. (60), the density follows from Poisson's equation as

$$n = n(0) \left[\frac{e^{r/\lambda_D} - e^{-r/\lambda_D}}{r/\lambda_D} - \frac{e^{a/\lambda_D} - e^{-a/\lambda_D}}{a/\lambda_D} \right] \Bigg/ \left[2 - \frac{e^{a/\lambda_D} - e^{-a/\lambda_D}}{a/\lambda_D} \right] \quad , \tag{64}$$

where we have introduced the Debye length according to

$$\lambda_D^2 = C^{-1} \tag{65}$$

and defined a as the beam edge.

Introducing the plasma frequency (on axis) by

$$w_p^2 = \frac{q^2 n(0)}{\varepsilon_o m \gamma^3 v^2} \quad , \tag{66}$$

we can write

$$\lambda_D^2 = \frac{\mu}{w_p^2} = \frac{\overline{x'^2}}{w_p^2} = \frac{v_{thermal}^2}{w_p^2} \tag{67}$$

in agreement with the usual definition of Debye length.

Equation (64) reveals the shielding behavior, which can be seen more easily by rewriting the expression, using $\lambda_D << a$ (avoiding $r = 0$)

$$n = n(0) \left[1 - \frac{e^{(r-a)/\lambda_D)}}{r/a} \right] \quad . \tag{68}$$

The density is thus uniform except for a sheath of thickness λ_D, where it drops to zero. Comparison of Eq. (68) [more accurately Eq. (64)] with Eq. (59) shows that the solution found is consistent with the series truncation as long as $|r - a| > \lambda_D$ and $\lambda_D << a$. Inside the boundary sheath the truncation is invalidated when approaching r = a, where the full solution cannot have a sharp edge.

For a waterbag (i.e., step-function) distribution defined as

$$f = \frac{n(0)}{\frac{4}{3}\pi(3\mu)^{3/2}}\,\theta\left[\frac{x'^2 + y'^2 + z'^2}{2} + \frac{k}{2}r^2 + \frac{q}{m\gamma^3 v^2}\phi - \frac{3}{2}\mu\right] , \tag{69}$$

we readily obtain the nonlinear equation

$$\nabla^2\phi = -\frac{q}{\varepsilon_o}\,n(0)\left[1 - \left(\frac{k}{2}r^2 + \frac{q}{m\gamma^3 v^2}\phi\right)\Big/\frac{3}{2}\mu\right]^{3/2} . \tag{70}$$

This can be expanded and, with the leading term, we obtain again Eq. (59); hence, the same density profiles as in Eqs. (64) and (68). The different velocity space profile of the waterbag distribution leads to $\mu = 5/3\langle x'^2\rangle_{r=0}$ (r = 0 denoting the local spread of x' on axis); however, we readily find from Eq. (69) that for $\lambda_D << a$, the local average of x'^2 is constant, except for the boundary sheath because of the smallness of the potential energy compared with $\frac{3}{2}\mu$. We thus replace $\langle x'^2\rangle_{r=0}$ by $\overline{x'^2}$ and obtain

$$\lambda_D^2 = \frac{m}{w_p^2} \approx \frac{5}{3}\,\frac{\overline{x'^2}}{w_p^2} . \tag{71}$$

The Debye length can be related to the more familiar betatron tune depression ν/ν_o if we use the harmonic betatron oscillation approximation. With $\nu_o^2 = k$ and Eqs. (1), (66), and (A-2), we find for the space-charge-depressed tune,

$$\nu^2 = \nu_o^2 - \frac{w_p^2}{3} , \tag{72}$$

and with $\overline{x'^2} \approx \nu^2\,\overline{x^2}$, we readily obtain for the Gaussian distribution

$$\frac{\lambda_D}{a} \approx \frac{n}{[15(\nu_o^2 - \nu^2)]^{1/2}} \approx \frac{1}{\sqrt{15}}\,\frac{\nu}{\nu_o} \tag{73}$$

and similarly, for the waterbag distribution,

$$\frac{\lambda_D}{a} \approx \frac{1}{3} \frac{\nu}{\nu_o} \quad . \tag{74}$$

B. Two-Dimensional Case

For the Gaussian distribution, we obtain in the same manner as before a Poisson equation [as in Eq. (58)], which after linearization gives again

$$\nabla^2\phi = - \frac{q}{\varepsilon_o} n(0) \left[1 - \left(\frac{k}{2} r^2 + \frac{q}{m\gamma^3 v^2} \phi\right)\Big/\mu\right] \quad . \tag{75}$$

With a substitution analogous to Eq. (60) (the 6 is replaced by 4, in this case), we find as solutions for $\tilde{\phi}$, the modified Bessel functions of zero order $I_o(\sqrt{C}\, r)$ and the density

$$n = n_o \left[1 - \frac{I_o(r/\lambda_D)}{I_o(a/\lambda_D)}\right] \quad , \tag{76}$$

where λ_D is again defined as in Eq. (67).

For the waterbag distribution in 2-D, we find that Eq. (75) holds exactly, a well-known result.[8] Hence, Eq. (76) applies exactly and can be used to construct an exact stationary distribution. The Debye length in this case is given by

$$\lambda_D^2 = \frac{\mu}{w_p^2} \approx 2 \frac{\overline{x'^2}}{w_p^2} \quad . \tag{77}$$

For large r/λ_D, we can use the asymptotic expansion

$$I_o(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left(1 + \frac{1}{8z} + \ldots\right) \tag{78}$$

and obtain

$$n = n_o \left[1 - \frac{e^{(r-a)/\lambda_D}}{\sqrt{r/a}}\right] \quad , \tag{79}$$

which reveals again the uniform profile, except for the boundary sheath of thickness λ_D.

We can express this again in terms of the tune depression by using Eq. (B-2) and deriving the 2-D equivalent of Eq. (72):

$$\nu^2 = \nu_o^2 - \frac{w_p^2}{2} \quad . \tag{80}$$

Similarly, we find for the Gaussian distribution

$$\frac{\lambda_D}{a} \approx \frac{1}{\sqrt{8}} \frac{\nu}{\nu_o} \quad , \tag{81}$$

and for the waterbag distribution

$$\frac{\lambda_D}{a} \approx \frac{1}{2} \frac{\nu}{\nu_o} \quad . \tag{82}$$

VI. APPLICATIONS

While we intend to evaluate the practical usefulness of our generalized emittance equations in a subsequent detailed study based on computer simulation, it will be useful here to outline a few basic relationships. From the structure of the equations, we attempt a classification of the following types of emittance growth:

A. Initial Mismatch Emittance Growth

An rms matched beam is (intrinsically) mismatched if the nonlinear field-energy term changes rapidly within a coherent oscillation period, which is comparable with the plasma period given by Eq. (66) for a beam close to the space-charge limit [in this limit, the plasma period is identical to $1/\sqrt{3}$ (3-D case) or $1/\sqrt{2}$ (2-D case) times the zero-current betatron period according to Eq. (72) or (80)]. The conversion of this field energy into emittance growth can be estimated from Eq. (28) if we assume that the rms envelopes remain nearly constant, an assumption that is certainly justified in a space-charge-dominated beam. We thus obtain for the 3-D case of a rotationally symmetric ellipsoid $(\overline{x^2} = \overline{y^2})$

$$2\,\frac{\Delta\varepsilon_\perp^2}{\overline{x^2}} + \frac{\Delta\varepsilon_z^2}{\overline{z^2}} \approx - \frac{32}{m\gamma^3 v^2 N}\,\Delta(W - W_u) \quad . \tag{83}$$

(The change in λ_3 gives only a small correction negligible here.) This expression can be rewritten if we use the third equation, Eq. (1), in smooth approximation, evaluating E_z from Eq. (A-2); hence, we obtain, with $\nu_{oz}^2 \equiv k_z$ and ν_z^2 the corresponding space-charge-depressed betatron tune:

$$\frac{Nq^2}{m\gamma^3 v^2 4\pi\varepsilon_o} = \nu_{oz}^2 - \nu_z^2\,\frac{a^2 c}{3f} \tag{84}$$

$\left[a^2 = 5\,\overline{x^2},\ c^2 = 5\,\overline{z^2} \text{ and } f \text{ from Eq. (A-13) or (A-14)}\right]$.

Using Eq. (5) for the initially upright phase ellipse and the harmonic betatron-oscillation approximation, we find, for the input emittance,

$$\varepsilon_\perp^2 = 16\,\overline{x^2}\;\overline{x'^2} = 16\,\nu_x^2\,\overline{x^2}^2 \tag{85}$$

and similar in z. We are thus able to express Eq. (83) in the two following forms:

$$2\,\frac{\Delta\varepsilon_\perp^2}{\varepsilon_\perp^2} + \frac{\nu_z^2\,\overline{z^2}}{\nu_x^2\,\overline{x^2}}\,\frac{\Delta\varepsilon_z^2}{\varepsilon_z^2} \approx - \left(\frac{\nu_{ox}^2}{\nu_x^2} - 1\right) \frac{2/3}{1 - f} \left(\frac{\overline{z^2}}{\overline{x^2}}\right)^{1/3} \Delta\,\frac{W - W_u}{w_1}$$

and (86)

$$2\,\frac{\nu_x^2\,\overline{x^2}}{\nu_z^2\,\overline{z^2}}\,\frac{\Delta\varepsilon^2}{\varepsilon_\perp^2} + \frac{\Delta\varepsilon_z^2}{\varepsilon_z^2} \approx - \left(\frac{\nu_{oz}^2}{\nu_z^2} - 1\right) \frac{1}{3f} \left(\frac{\overline{x^2}}{\overline{z^2}}\right)^{2/3} \Delta\,\frac{W - W_u}{w_1} \quad ,$$

where we have introduced w_1 as normalization constant

$$w_1 = \frac{N^2 q^2}{40\pi\varepsilon_o \bar{R}_o} \quad , \tag{87}$$

which is the field energy calculated inside a uniform spherical bunch of the same volume; hence, its rms radius is given by

$$\overline{R}_o^3 = \overline{x^2}\,(\overline{z^2})^{1/2} = \frac{a^2 c}{5^{3/2}} \quad . \tag{88}$$

The equations in Eq. (86), with tunes and rms envelopes evaluated from the initial conditions, allow us to calculate emittance change from the change of the nonlinear field energy. We observe that our equations do not indicate how the total emittance change is distributed into the transverse and the horizontal planes. If we make the assumption--in a thermodynamic sense--that excess nonlinear field energy increases equally the "temperatures" (given by $\overline{x'^2}$, $\overline{z'^2}$) in each degree of freedom, then the terms $\Delta\varepsilon_\perp^2/\overline{x^2}$ and $\Delta\varepsilon_z^2/\overline{z^2}$ in Eq. (83) are approximately equal, and we readily find

$$\frac{\Delta\varepsilon_\perp^2}{\varepsilon_\perp^2} \approx -\frac{1}{3}\left(\frac{\nu_{ox}^2}{\nu_x^2} - 1\right)\frac{2/3}{1-f}\left(\frac{\overline{z^2}}{\overline{x^2}}\right)^{1/3} \Delta\,\frac{W - W_u}{w_1}$$

and (89)

$$\frac{\Delta\varepsilon_z^2}{\varepsilon_z^2} \approx -\frac{1}{3}\left(\frac{\nu_{oz}^2}{\nu_z^2} - 1\right)\frac{1}{3f}\left(\frac{\overline{x^2}}{\overline{z^2}}\right)^{2/3} \Delta\,\frac{W - W_u}{w_1} \quad .$$

To estimate the actual change of nonlinear field energy, we can use the result of Sec. V, according to which

$$\frac{W - W_u}{w_1} \rightarrow 0 \tag{90}$$

for $\lambda_D/a \rightarrow 0$, i.e., $\nu/\nu_o \rightarrow 0$. Hence, For $\nu/\nu_o \ll 1$ in the longitudinal and transverse plane, the stationary nonlinear field energy is small; it can be calculated for a spherical bunch from Eq. (68). The emittance growth resulting from a strongly mismatched density profile can thus readily be evaluated; a parabolic profile for a spherical bunch yields, for instance,

$$\left(\frac{W - W_u}{w_1}\right)_{\text{parabolic}} = 0.0368 \tag{91}$$

For this spherically symmetric case, we can thus immediately conclude from Eqs. (86) and (89) that

$$\frac{\Delta\varepsilon^2}{\varepsilon^2} \approx -\frac{1}{3}\left(\frac{\nu_o^2}{\nu^2} - 1\right) \Delta \frac{W - W_u}{w_1} \tag{92}$$

and

$$\frac{\Delta\varepsilon^2}{\varepsilon^2} \lesssim \frac{0.0368}{3}\left(\frac{\nu_o^2}{\nu^2} - 1\right) , \tag{93}$$

where we have obtained an upper limit by neglecting the final (positive) nonlinear field energy. For more accuracy, we could do an iteration by using Eq. (93) to determine a new ν/ν_o and λ_D/a, according to Eqs. (74) and (85); thus, we use Eqs. (64) or (68) to calculate the nonlinear field energy for the corrected shielded stationary distribution and use this as the final value in $\Delta[(W - W_u)/w_1]$ of Eq. (92). We note that it is only for $\nu/\nu << 1$ that Eq. (92) predicts a noticeable emittance growth; therefore, it is appropriate to make use of the shielding concept in this discussion.

The 2-D analogue of Eq. (86) is derived in similar fashion as

$$\frac{\Delta\varepsilon_x^2}{\varepsilon_x^2} + \frac{\nu_y^2\,\overline{y^2}}{\nu_x^2\,\overline{x^2}}\,\frac{\Delta\varepsilon_y^2}{\varepsilon_y^2} \approx -\frac{\overline{x^2}^{1/2} + \overline{y^2}^{1/2}}{2\overline{y^2}^{1/2}}\left(\frac{\nu_{ox}^2}{\nu_x^2} - 1\right) \Delta \frac{W - W_u}{w_o} \tag{94}$$

(or with x,y interchanged).

Emittance increase in each phase plane can again be calculated if the above thermodynamic argument of equidistribution is used, yielding

$$\frac{\Delta\varepsilon_x^2}{\varepsilon_x^2} \approx -\frac{1}{2}\,\frac{\overline{x^2}^{1/2} + \overline{y^2}^{1/2}}{2\,\overline{y^2}^{1/2}}\left(\frac{\nu_{ox}^2}{\nu_x^2} - 1\right) \Delta \frac{W - W_u}{w_o} \tag{95}$$

and also for ε_y with x,y interchanged.

The round beam limit yields

$$\frac{\Delta\varepsilon^2}{\varepsilon^2} = -\frac{1}{2}\left(\frac{\nu_o^2}{\nu^2} - 1\right) \Delta \frac{W - W_u}{w_o} , \tag{96}$$

which is in agreement with the result of Ref. 1. For a parabolic profile, we have[1] $(W - W_u)/w_o = 0.0224$; hence (ignoring again the final nonlinear field energy),

$$\frac{\Delta\varepsilon^2}{\varepsilon^2} \lesssim \frac{0.0224}{2} \left(\frac{\nu_o^2}{\nu^2} - 1 \right) \quad . \tag{97}$$

Comparing Eq. (97) with Eq. (93), we see that each degree of freedom gains about the same emittance increase from a parabolic profile in 2-D or 3-D.

We observe that, for a parabolic nonround beam, $(W - W_u)/w_o$ has the same value independent of the ellipticity, which then can be used in Eq. (95). (See App. D.) A Gaussian profile (truncated at four standard deviations) yields[1] the much larger value of $(W - W_u)/w_o = 0.154$.

Note that we have made use of the smooth approximation to derive the mismatch emittance-growth formulae; we expect however, that they are also valid in periodic focusing if the average envelope is used.

B. Emittance Transfer

Computer simulation in 2-D and 3-D beams has shown[5] that emittance transfer can happen in intense beams with strong anisotropy of temperature, (divergence). This transfer was accompanied by nonuniform density oscillations (at about the plasma frequency). Our emittance equations provide a framework to study this mechanism as well. We observe that emittance transfer occurs at a much slower time scale than the initial mismatch; hence, the latter has to be calculated first to provide the correct initial condition for emittance transfer.

While we cannot obtain from Eqs. (28) or (41) the explicit time dependence of individual emittances, we can show in the following discussion that an approximate invariant exists, which is of a practical value. In App. E, we show that the left-hand side of Eq. (28) can be expressed explicitly as a derivative in s if we use the envelope equations, Eqs. (29) and (30), in smooth approximation:

$$\frac{2}{x^2} \frac{d}{ds} \varepsilon_\perp^2 + \frac{1}{z^2} \frac{d}{ds} \varepsilon_z^2 \approx 2 \frac{d}{ds} \left[2 \frac{\varepsilon_\perp^2}{x^2} + \frac{\varepsilon_z^2}{z^2} + \frac{8}{Nm\gamma^3 v^2} W_u \right] \quad . \tag{98}$$

The right-hand side of Eq. (28) is oscillatory and thus yields a negligible integrated contribution (it is given by the change of $W - W_u$, which is negligible because we assume here that the beam has been properly matched initially). The approximate invariant is then

$$2 \frac{\varepsilon_\perp^2}{\overline{x^2}} + \frac{\varepsilon_z^2}{\overline{z^2}} + \frac{8}{Nm\gamma^3 v^2} W_u \approx \text{const} \tag{99}$$

or, with Eq. (5),

$$\frac{1}{32} \Delta \left(2 \frac{\varepsilon_\perp^2}{\overline{x^2}} + \frac{\varepsilon_z^2}{\overline{z^2}} \right) \approx \Delta \left(\frac{2\overline{x'^2} + \overline{z'^2}}{2} \right) \approx - \frac{1}{4} \frac{\Delta W_u}{Nm\gamma^3 v^2} \quad . \tag{100}$$

A comparison with the energy expression in Eq. (17) shows the physical meaning of this invariant: 1/4 of the change of the uniform field energy goes into the thermal energy of the beam, whereas the remaining 3/4 must go into the potential energy (with respect to the external focusing force).

In the 2-D case, the corresponding expressions are (see App. E)

$$\frac{\varepsilon_x^2}{\overline{x^2}} + \frac{\varepsilon_y^2}{\overline{y^2}} + \frac{16}{Nm\gamma^3 v^2} W_u \approx \text{const} \tag{101}$$

and

$$\Delta \frac{\overline{x'^2} + \overline{y'^2}}{2} \approx - \frac{1}{2} \frac{\Delta W_u}{Nm\gamma^3 v^2} \quad , \tag{102}$$

indicating that 1/2 of the uniform field-energy change goes into the potential energy, and the other 1/2 into the thermal energy.

In practice, we expect that, for high-current beams, the change in uniform field energy is small because the envelopes change only a little in a space-charge-dominated beam; hence, we may assume as a first-order estimate in 3-D that

$$2 \Delta \left(\frac{\varepsilon_\perp}{\overline{x^2}} \right) \approx - \Delta \left(\frac{\varepsilon_z^2}{\overline{z^2}} \right) \tag{103}$$

and

$$2\,\Delta\,\overline{x'^2} \approx -\,\Delta\,\overline{z'^2}\ ;$$

and in 2-D,

$$\Delta\left(\frac{\varepsilon_x}{\overline{x^2}}\right) \approx -\,\Delta\left(\frac{\varepsilon_y}{\overline{y^2}}\right)$$

and (104)

$$\Delta\,\overline{x'^2} \approx -\,\Delta\,\overline{y'^2}\ .$$

These estimates imply that the total thermal energy or divergence remains roughly constant during an emittance exchange, which agrees well with results from previous computer simulation.[5] We note that the nonlinear field energy does not appear in the approximate invariants defined in Eqs. (99) and (101), yet the nonlinear field energy plays a crucial role as the actual mechanism that drives the exchange dynamically, either by a coherent instability[5] or by single-particle resonance. We also note that, in computer simulation,[5] the presence of continuous or periodic focusing does not change the results on emittance transfer, as long as structure resonances are avoided (see next section). This justifies the use of the smooth approximation model here. Finally, we observe that there is actually a sufficiently strong imbalance of thermal energy, which is necessary for emittance transfer to occur;[5] hence, Eqs. (103) and (104) also provide an upper limit for the emittance growth in the initially "cold" plane.

C. Structure Resonances in Periodic Focusing

Multiplying Eq. (28) by $\overline{x^2}$, $\overline{y^2}$, or $\overline{z^2}$, we recognize the possibility that the corresponding emittance (possibly also the other two) increases with time if, because of some coherent oscillation, the nonlinear field energy contains the frequency of oscillation of the envelope $\overline{x^2}$ (or, respectively, $\overline{y^2}$ or $\overline{z^2}$) as induced by the periodically varying focusing force. In such a case of "structure resonance,"[7] we could derive an upper bound for the emittance increase with time, if we knew an upper bound to the nonlinear field energy. Obviously, this would require us to determine the eigenmodes of oscillation from the linearized Vlasov equation (as in Ref. 7), which is beyond the scope of this work.

Nonetheless, we can derive a relationship, which gives some general insight and the time scale for the emittance growth formed by a structure resonance.

To this end we assume, for simplicity, a round beam in periodic solenoidal focusing; hence, Eq. (43) applies. The squared rms envelope is assumed to contain as leading harmonic the periodicity of the focusing channel ($\omega = 2\pi/L$, L focusing period):

$$\overline{x^2}(s) = \hat{\overline{x^2}} + \delta\overline{x^2} \sin(\omega s) \quad , \tag{105}$$

where we ignore all other harmonics. Likewise, we assume the same harmonic within $U = \dfrac{W - W_u}{w_o}$ (phase shifted by 90°):

$$U = \hat{U} + \delta U \cos(\omega s) \tag{106}$$

and find a nonoscillating term for the emittance change

$$\frac{d}{ds} \varepsilon^2 \approx K\, \delta\overline{x^2}\, \omega\, \delta U \quad . \tag{107}$$

In the smooth approximation, we replace K by

$$K = 4 \frac{\hat{\varepsilon}^2}{\overline{x^2}} \left(\frac{\nu_o^2}{\nu^2} - 1 \right) \tag{108}$$

and integrate Eq. (107) to yield

$$\frac{\varepsilon}{\varepsilon_o} \approx \left[1 + \frac{\pi}{2} \left(\frac{\nu_o^2}{\nu^2} - 1 \right) \frac{\delta\overline{x^2}}{\hat{\overline{x^2}}}\, \delta U\, \frac{s}{L} \right]^{1/2} \quad . \tag{109}$$

A practical difficulty lies with estimating δU, which can develop exponentially from an infinitesimal noise level during the early stage of the resonance. We recognize, however, that the emittance growth is most significant after δU has come close to its maximum level, which is typically of the order of 10^{-1} (see Ref. 1 for calculations of U for different profiles). We then conclude from Eq. (109) that the rate of emittance growth during the first doubling of emittance is about proportional to $\frac{\nu_o^2}{\nu^2} - 1$ and to the relative envelope modulation, thereafter to the square root of these quantities.

Structure resonances are avoided, in practice, for systems with $\sigma_o = 60°$. A general discussion of the conditions for structure resonances is found in Refs. 7 and 9.

VII. CONCLUSIONS

We have shown that the generalized emittance equations derived here are of basic importance to understanding emittance growth and to evaluating it quantitatively on different time scales. These equations allow us to predict rapid mismatch emittance growth if the concept of Debye shielding in stationary distributions is applied. For the slower process of emittance transfer, we have suggested an approximate invariant, which applies regardless of the actual coupling process whether coherent or incoherent. A future systematic study of the conditions for emittance transfer by numerical simulation will be very helpful to further deepen our understanding. A further task will be to incorporate the formulary derived here and that obtained by numerical simulation into high-current linac design procedures.

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APPENDIX A

THREE-DIMENSIONAL FIELD ENERGY CALCULATION

From Eq. (14), we obtain for the field energy in a volume V by partial integration,

$$W = \frac{1}{2} \iiint \phi \, \rho \, dx \, dy \, dz - \frac{\varepsilon_o}{2} \iint_S \phi \, E_n d\sigma \quad . \tag{A-1}$$

Assuming a uniformly charged ellipsoid with rotational symmetry around the z-axis, we can write for the space-charge potential inside the beam

$$\phi_i = - \frac{\rho_o}{2\varepsilon_o} \left[(x^2 + y^2) \frac{1-f}{2} + z^2 f \right] \quad , \tag{A-2}$$

with f yet to be determined by the condition of continuous potential and field across the boundary of the ellipsoid given by

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{c^2} = 1 \quad .$$

For an oblate spheroid* (a > c), we introduce the variables u, v, and φ according to

$$x = \alpha \sin u \cosh v \cos \varphi \ ,$$

$$y = \alpha \sin u \cosh v \sin \varphi \ , \text{ and} \tag{A-3}$$

$$z = \alpha \cos u \sinh v$$

with $\alpha^2 = a^2 - c^2$ and the boundary given by $v = v_o$; that is, $a = \alpha \cosh v_o$ and $c = \alpha \sinh v_o$.

Introducing

$\xi = \cos u$ ("angular" variable)
and (A-4)
$\eta = \sinh v$ ("radial" variable) .

*The proof for a prolate spheroid (c < a) is analogous and will be omitted here.

Laplace's equation for the exterior solution can be written as[10]

$$\nabla^2\phi_e = \frac{1}{\alpha^2(\eta^2+\xi^2)}\left\{\frac{\partial}{\partial\xi}\left[\left(1-\xi^2\right)\frac{\partial\phi}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[\left(1+\eta^2\right)\frac{\partial\phi}{\partial\eta}\right]\right\} = 0 \quad . \tag{A-5}$$

This allows a separation of variables for the angular and the radial part; a general solution, which is regular and vanishes at infinity, can be expanded as[10]

$$\phi_e = A_o + \sum_\nu B_\nu P_\nu(\xi)\, Q_\nu(i\eta) \tag{A-6}$$

with $P_\nu(\xi)$ Legendre polynomials and $Q_\nu(i\eta)$ Legendre functions of the second kind with imaginary argument. The interior solution can be rewritten in terms of Legendre polynomials in the new variable ξ:

$$\phi_i = -\frac{\rho_o\alpha^2}{6\varepsilon_o}\left[\left(\eta^2+1-f\right)P_o+\left(3\eta^2 f-1-\eta^2+f\right)P_2\right] \quad . \tag{A-7}$$

Because of the orthogonality of Legendre polynomials and the requirement of continuity of ϕ on the boundary, only B_o and B_2 in the expansion of Eq. (A-6) are different from zero. We thus obtain, for $\eta_o = c/\alpha$, two equations for the continuity of ϕ:

$$A_o + B_o \cot^{-1}\eta_o = -\frac{\rho_o\alpha^2}{6\varepsilon_o}\left(\eta_o^2+1-f\right) \tag{A-8}$$

and

$$B_2\left[\left(3\eta_o^2+1\right)\cot^{-1}\eta_o - 3\eta_o\right] = -\frac{\rho_o\alpha^2}{6\varepsilon_o}\left(3\eta_o^2 f-1-\eta_o^2+f\right) \quad , \tag{A-9}$$

where we have used $Q_o(i\eta) = -i\cot^{-1}\eta$ and $Q_2(i\eta) = i/2[(3\eta^2+1)\cot^{-1}\eta - 3\eta]$. Likewise, we obtain for continuous $\partial\phi/\partial\eta$

$$-\frac{B_o}{1+\eta_o^2} = -\frac{\rho_o\alpha^2}{3\varepsilon_o}\eta_o \tag{A-10}$$

and

$$B_2\left(6\eta_o\cot^{-1}\eta_o - \frac{3\eta_o^2+1}{1+\eta_o^2} - 3\right) = -\frac{\rho_o\alpha^2}{3\varepsilon_o}\left(3\eta_o f - \eta_o\right) \quad , \tag{A-11}$$

where we have used (see Ref. 10, p. 145)

$$\frac{d}{d\eta} \cot^{-1} \eta = - \frac{1}{1 + \eta^2} \quad . \tag{A-12}$$

We thus have four equations for the unknowns f, A_o, B_o, and B_2, which result in

$$f = \left(1 + \eta_o^2\right)\left(1 - \eta_o \cot^{-1} \eta_o\right) \tag{A-13}$$

or, with $p \equiv c/a$ $(p < 1)$, in

$$f = \frac{1}{1 - p^2} - \frac{p}{(1 - p^2)^{3/2}} \cos^{-1} p \quad . \tag{A-14}$$

The equivalent expression for $p > 1$ (prolate spheroid) is found as

$$f = \frac{p \cosh^{-1} p}{(p^2 - 1)^{3/2}} - \frac{1}{p^2 - 1} \tag{A-15}$$

with $\cosh^{-1} p = \ln[p + (p^2 - 1)^{1/2}]$. In the near spherical limit $(0.8 \lesssim p \lesssim 5)$, one finds the approximate expression

$$f \approx \frac{1}{3p} \quad , \tag{A-16}$$

with $f = 1/3$ for a sphere.

These expressions for f are in agreement with Ref. 11. The remaining unknowns are

$$A_o = - \frac{\rho_o \alpha^2}{3\varepsilon_o} \left[\eta_o\left(1 + \eta_o^2\right) \cot^{-1} \eta_o + \frac{\eta_o^2}{2} + \frac{1 - f}{2}\right] \quad , \tag{A-17}$$

$$B_o = i \frac{\rho_o \alpha^2}{3\varepsilon_o} \eta_o\left(1 + \eta_o^2\right) \quad , \tag{A-18}$$

and

$$B_2 = i \frac{\rho_o \alpha^2}{3\varepsilon_o} \frac{3\eta_o^2 f - 1 - \eta_o^2 + f}{\left(3\eta_o^2 + 1\right)\cot^{-1} \eta_o - 3\eta_o} \quad . \tag{A-19}$$

Here we are interested in the asymptotic expression for ϕ_e at large distances, noting that

$$\eta \to \frac{r}{\alpha} \quad \text{as} \quad (\eta \to \infty) \ . \tag{A-20}$$

With the asymptotic expansion of $\cot^{-1}\eta$ derived from Eq. (A-12),

$$\cot^{-1}\eta = \frac{1}{\eta} - \frac{1}{3\eta^3} + \frac{1}{5\eta^5} - \dots \ , \tag{A-21}$$

we readily find the asymptotic behavior $Q_o \sim 1/\eta$ and $Q_2 \sim 1/\eta^3$, and thus

$$\phi_{ex} = A_o - iB_o \cot^{-1}\eta + 0\left(\frac{1}{\eta^3}\right) \tag{A-22}$$

or, with $\rho_o = \dfrac{Nq}{\frac{4}{3}a^2c\pi}$,

$$\phi_{ex} = \frac{Nq\alpha^2}{4\pi\varepsilon_o a^2 c}\left[\eta_o\left(1 + \eta_o^2\right)\left(\frac{1}{\eta} - \cot^{-1}\eta_o\right) - \frac{1}{2}\left(\eta_o^2 + 1 - f\right)\right] \ . \tag{A-23}$$

To evaluate W from Eq. (A-1), we need the asymptotic expression for $E_n = -\partial\phi/\partial r$, which results from Eq. (A-23) as

$$E_n = \frac{Nq}{4\pi\varepsilon_o r^2} + 0\,\frac{1}{r^4} \ . \tag{A-24}$$

The leading term of E_n is identical with the field of a point charge at the origin, as expected.

Using Eqs. (A-2), (A-13), (A-18) and (A-24), we obtain from Eq. (A-1) that the field energy W_u of a uniformly charged ellipsoid calculated within a large sphere of radius R ($\eta \to R/\alpha$) is given as

$$W_u = \frac{N^2q^2}{40\pi\varepsilon_o}\left[\frac{6}{c}\left(1 - f + \frac{c^2}{a^2}f\right) - \frac{5}{R}\right] \ . \tag{A-25}$$

Assuming that the semiaxes a, c, and thus f(c/a) are functions of s, we are able to calculate dW_u/ds. To this end, it is convenient to use Eq. (A-13) and write

$$W_u = \frac{N^2q^2}{40\pi\varepsilon_o}\left[\frac{6}{c}\,\eta_o \cot^{-1}\,\eta_o - \frac{5}{R}\right] \quad . \tag{A-26}$$

With Eq. (A-12), we find

$$\frac{dW_u}{ds} = -\frac{3N^2q^2}{20\pi\varepsilon_o}\left[\frac{dc}{ds}\,\frac{f}{a^2} + \frac{da}{ds}\,\frac{1-f}{ac}\right] \quad . \tag{A-27}$$

APPENDIX B

TWO-DIMENSIONAL FIELD ENERGY CALCULATION

Analogous to Eq. (A-1), we have

$$W = \frac{1}{2} \iint \phi\rho \; dx \; dy - \frac{\varepsilon_o}{2} \int_s \phi \; E_n \; d\sigma \quad , \tag{B-1}$$

for the field energy per unit length and within a cross-sectional area bounded by s. For the potential inside a beam with uniform density and semiaxes a,b, we have the well-known expression

$$\phi_i = - \frac{\rho_o}{2\varepsilon_o} \frac{bx^2 + ay^2}{a + b} \quad , \tag{B-2}$$

from which we readily calculate

$$\frac{1}{2} \iint \phi\rho = - \frac{q^2 N^2}{16\pi\varepsilon_o} \quad . \tag{B-3}$$

To derive the exterior solution, we introduce elliptic coordinates according to

$$x = \frac{\alpha}{2} \cosh \mu \cos \theta \quad ,$$

$$y = \frac{\alpha}{2} \sinh \mu \sin \theta \quad , \text{ and} \tag{B-4}$$

$$\left(\frac{\alpha}{2}\right)^2 = a^2 - b^2 \quad ,$$

with $a = \frac{\alpha}{2} \cosh \mu_o$ and $b = \frac{\alpha}{2} \sinh \mu_o$. With these coordinates, we obtain

$$\phi_i = - \frac{\rho_o ab}{4\varepsilon_o (a + b)} \left(\frac{\alpha}{2}\right)^2 \left[\frac{\cosh^2 \mu}{a} + \frac{\sinh^2 \mu}{b} + \left(\frac{\cosh^2 \mu}{a} - \frac{\cosh^2 \mu}{b}\right) \cos 2\theta\right] \quad . \tag{B-5}$$

For the exterior solution, we use the Green's function expansion in elliptic coordinates and make the substitution[12])

$$\phi_e = B_o \mu + B_1 \ln \frac{\alpha}{4} + B_2 e^{-2\mu} \cos 2\theta \quad . \tag{B-6}$$

Matching potential and field at the boundary given by $\mu = \mu_o$ results in

$$\phi_e = - \frac{\rho_o ab}{2\ \varepsilon_o} \left[\mu - \mu_o + \frac{1}{2} + \frac{1}{2} \frac{a - b}{a + b} e^{-2(\mu - \mu_o)} \cos 2\theta \right] . \tag{B-7}$$

At large distances from the beam, the angle-dependent term can be neglected, and we have, with $\rho_o = N/(ab\pi)$,

$$\phi_e = - \frac{Nq}{2\pi\varepsilon_o} \left(\ln \frac{2r}{a + b} + \frac{1}{2} \right) , \tag{B-8}$$

where we have used the asymptotic behavior $\mu \to \ln(4r/\alpha)$. The faraway E_n becomes

$$E_n = \frac{Nq}{2\pi\varepsilon_o r} , \tag{B-9}$$

which is the field from a line charge.

With Eq. (B-1), we obtain for the uniform-beam field energy within a circle of radius R

$$W_u = \frac{N^2 q^2}{16\pi\varepsilon_o} \left(1 + 4 \ln \frac{2R}{a + b} \right) . \tag{B-10}$$

APPENDIX C

ONE-DIMENSIONAL FIELD ENERGY CALCULATION

The potential of a uniformly charged sheet of thickness $\pm c$ is readily found as

$$\phi_i = - \frac{Nq}{4\varepsilon_o c} z^2 \qquad (|z| \lesssim c) \tag{C-1}$$

$$\phi_e = - \frac{Nq}{4\varepsilon_o} c(1 + 2 \frac{z - c}{c}) \qquad (|z| \gtrsim c) \tag{C-2}$$

and we obtain for the field energy within $|z| \leqq L$

$$W_u = \frac{N^2q^2}{\varepsilon_o} \frac{L}{4} - \frac{c}{6} \quad . \tag{C-3}$$

APPENDIX D
TWO-DIMENSIONAL PARABOLIC PROFILE BEAM

We assume a parabolic density profile

$$n = \frac{2N}{\pi ab}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \quad . \tag{D-1}$$

The interior electric field is[8]

$$E_x = \frac{2\ Nq}{\pi\varepsilon_o}\left[\frac{x}{a(a+b)} - \frac{x}{a(a+b)^2}\left(\frac{x^2}{3a^2}\ (2a+b) + \frac{y^2}{b}\right)\right] \tag{D-2}$$

(E_y with x,y and a,b interchanged), with the respective potential as

$$\phi_i = -\frac{Nq}{\pi\varepsilon_o(a+b)}\left[\frac{x^2}{a} + \frac{y^2}{b} - \frac{1}{6a^3}\,\frac{2a+b}{a+b}\,x^4 - \frac{1}{ab(a+b)}\,x^2y^2 - \frac{1}{6b^3}\,\frac{2b+a}{a+b}\,y^4\right] . \tag{D-3}$$

For the exterior potential, we use the Green's function expansion in elliptic coordinates[12] (see also App. B):

$$\phi_e = B_o\mu + B_1\ \ln\frac{\alpha}{4} + B_2e^{-2\mu}\cos 2\theta + B_4e^{-4\mu}\cos 4\theta \quad . \tag{D-4}$$

From the requirement of continuity and continuous derivatives, we find for the leading term

$$\phi_e = \frac{Nq}{2\pi\varepsilon_o}\left[\mu - \left(\mu_o - \frac{3}{4}\right)\right] \quad . \tag{D-5}$$

The terms with cos 2θ, cos 4θ decrease exponentially at large distance because for large μ

$$\mu - \mu_o \to \ln\frac{2r}{a+b} \quad . \tag{D-6}$$

We thus find for large r

$$\phi_e \approx -\frac{Nq}{2\pi\varepsilon_o}\left(\ln\frac{2r}{a+b} + \frac{3}{4}\right) , \tag{D-7}$$

and the normal component

$$E_n = -\frac{\partial \phi}{\partial r} = \frac{Nq}{2\pi\varepsilon_o r} \quad . \tag{D-8}$$

With Eq. (B-1), we obtain

$$W = \frac{N^2 q^2}{16\pi\varepsilon_o}\left[\frac{11}{6} - 4 \ln \sqrt{6} + 4 \ln \frac{R}{\tilde{x}}\right] \quad , \tag{D-9}$$

where we introduce the rms envelope

$$\tilde{x} = a/\sqrt{6} \quad . \tag{D-10}$$

Using Eq. (B-10)(with $a = 2\,\tilde{x}$), we readily find

$$\frac{W - W_u}{w_o} = \frac{5}{6} - 4 \ln \frac{\sqrt{6}}{2} \approx 0.0224 \quad . \tag{D-11}$$

APPENDIX E
INVARIANT EXPRESSION FOR EMITTANCE TRANSFER

The left-hand side of Eq. (28) can be rewritten as

$$\frac{2}{\overline{x^2}}\frac{d}{ds}\varepsilon_\perp^2+\frac{1}{\overline{z^2}}\frac{d}{ds}\varepsilon_z^2=2\frac{d}{ds}\left(\frac{\varepsilon_\perp^2}{\overline{x^2}}\right)+\frac{d}{ds}\left(\frac{\varepsilon_z^2}{\overline{z^2}}\right)$$

$$+\,2\frac{\varepsilon_\perp^2}{\overline{x^2}^2}\frac{d}{ds}\overline{x^2}+\frac{\varepsilon_z^2}{\overline{z^2}^2}\frac{d}{ds}\overline{z^2}\quad . \tag{E-1}$$

To evaluate the second term on the right-hand side, we use the envelope equations, Eqs. (29) and (30), in smooth approximation (k_x, k_z constant and $d^2\tilde{x}/ds$, $d^2\tilde{z}/ds$ negligible), obtaining

$$\frac{\varepsilon_\perp^2}{\tilde{x}^4}\approx 16\,k_x-16\,\frac{h_x}{\tilde{z}\tilde{x}^2}\,L \tag{E-2}$$

$$\frac{\varepsilon_z^2}{\tilde{x}^4}\approx 16\,k_z-16\,\frac{h_z}{\tilde{x}\tilde{z}^2}\,L \tag{E-3}$$

with $L = Nq^2/(20\sqrt{5}\,\pi\varepsilon_0 m\gamma^3v^2)$ and $\tilde{x}^2=\overline{x^2}$, $\tilde{z}^2=\overline{z^2}$. We thus have ignored the rapid flutter of $\tilde{x}$, $\tilde{z}$ and only consider slow changes due to ε , ε_z changing on a slow time-scale. Using Eqs. (E-2), (E-3), and (A-24), we find:

$$2\,\frac{\varepsilon_\perp^2}{\tilde{x}^4}\frac{d}{ds}\tilde{x}^2+\frac{\varepsilon_z^2}{\tilde{z}^4}\frac{d}{ds}\tilde{z}^2\approx 16\left(2\,k_x\frac{d}{ds}\tilde{x}^2+k_z\frac{d}{ds}\tilde{z}^2+\frac{2}{Nm\gamma^3v^2}\frac{d}{ds}W_u\right)\quad , \tag{E-4}$$

where we have replaced λ_3 by 1. The first term on the r.h.s. of this equation is the derivative of the potential energy, which we can express, by using again Eqs. (E-2), (E-3), and (A-27), as

$$16\left(2\,k_x\frac{d}{ds}\tilde{x}^2+k_z\frac{d}{ds}\tilde{z}^2\right)\approx\frac{d}{ds}\left(2\,\frac{\varepsilon_\perp^2}{\tilde{x}^2}+\frac{\varepsilon_z^2}{\tilde{x}^2}-\frac{16}{Nm\gamma^3v^2}\frac{d}{ds}W_u\right)\,. \tag{E-5}$$

We thus obtain from Eq. (E-1):

$$\frac{2}{\overline{x^2}} \frac{d}{ds} \varepsilon_\perp^2 + \frac{1}{\overline{z^2}} \frac{d}{ds} \varepsilon_z^2 \approx 2 \frac{d}{ds} \left[\frac{2\varepsilon_\perp^2}{\overline{x^2}} + \frac{\varepsilon_z^2}{\overline{z^2}} + \frac{8}{Nm\gamma^3 v^2} W_u \right] . \tag{E-6}$$

For 2-D beams, we see from Eqs. (44) and (45) that Eq. (E-5) is replaced by

$$16 \left(2\, k_x \frac{d}{ds} \tilde{x}^2 + k_y \frac{d}{ds} \tilde{y}^2 \right) \approx \frac{d}{ds} \left(\frac{\varepsilon_x^2}{\tilde{x}^2} + \frac{\varepsilon_y^2}{\tilde{y}^2} \right) \quad ; \tag{E-7}$$

hence, we find

$$\frac{1}{\overline{x^2}} \frac{d}{ds} \varepsilon_x^2 + \frac{1}{\overline{y^2}} \frac{d}{ds} \varepsilon_y^2 \approx 2 \frac{d}{ds} \left[\frac{\varepsilon_x^2}{\overline{x^2}} + \frac{\varepsilon_y^2}{\overline{y^2}} + \frac{16}{Nm\gamma^3 v^2} W_u \right] \quad , \tag{E-8}$$

which yield the invariants in Sec. VI.2.

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